Supplemental Notes

EE 503 Week 11 Dr. Franzke

HW # 11

1. Gubner 6.27 – 6.28
2. Leon-Garcia 4.102 – 4.106
   5.99 – 5.102
   7.23 – 7.29
   8.39 – 8.41

Topics

1. Central Limit Theorem (CLT)

- $Z_n \xrightarrow{d} Z \sim N(0,1)$ random sample
- $(1-\alpha)$ % C.I. for μ_X : $\overline{X}_n \pm z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$

$$Z_n = STD(\overline{X}_n) = STD\left(\sum_{k=1}^{n} X_k\right)$$

Recall for random samples: $X_1, X_2, \ldots$ $(\sigma^2 < \infty)$

$$\text{LLN:} \quad \bar{X}_n \xrightarrow{\text{mopd}} \mu_x$$

$$\text{CLT:} \quad Z_n \xrightarrow{d} Z \sim N(0,1)$$

for $Z_n = \text{STD}(\bar{X}_n) = \dfrac{\bar{X}_n - \mu_x}{\sigma_x/\sqrt{n}}$

$$= \text{STD}\left(\sum_{k=1}^{n} X_k\right)$$

$$= \frac{\sum_{k=1}^{n} X_k - n \cdot \mu_x}{\sqrt{n} \cdot \sigma_x}$$

$\therefore$ for "large n" $(n \geq 30)$:

$$\bar{X}_n \overset{d}{\approx} N\left(\mu_x, \frac{\sigma_x^2}{n}\right) \quad \text{and} \quad \sum_{k=1}^{n} X_k \sim N(n \cdot \mu_x, n \cdot \sigma_x^2)$$

$$\therefore \sqrt{n}(\bar{X}_n - \mu_x) = \frac{\sigma}{\sigma} \cdot \sqrt{n}(\bar{X}_n - \mu_x)$$

$$= \sigma \cdot \left(\frac{\bar{X}_n - \mu_x}{\sigma_x/\sqrt{n}}\right)$$

$$\overset{d}{\approx} \sigma \cdot Z$$

$$\boxed{\therefore \sqrt{n}(\bar{X}_n - \mu_x) \overset{d}{\approx} N(0, \sigma_x^2)}$$

For iid k-vectors $X_k \sim K_{xx}$ ← positive semi-definite

$$\sqrt{n}(\underset{k\times 1}{\bar{X}_n} - \underset{k\times 1}{\mu_x}) \overset{d}{\approx} N(\underset{k\times 1}{0}, \underset{k\times k}{K_{xx}}) \quad \text{and} \quad \underset{k\times 1}{\bar{X}_n} \overset{d}{\approx} N(\underset{k\times 1}{\mu_x}, \underset{k\times k}{\tfrac{1}{n} K_{xx}})$$

How good is the CLT approximation?

The Berry-Essen Theorem says that the approximation error falls off with $\frac{1}{\sqrt{n}}$ if third moment exists.

Thrm: (Berry-Essen) $\exists c > 0 \ \forall x \ \forall n.$
universal constant

$$\boxed{\left| F_n(x) - \Phi(x) \right| \leq \frac{c \cdot E[|X-\mu_x|^3]}{\sqrt{n} \cdot \sigma_x^3}} \quad \propto \frac{1}{\sqrt{n}}.$$

if iid $X_1, X_2, \ldots$ and $E[|X-\mu_x|^3] < \infty$

and $F_n(x) = P\left(\frac{\bar{X}_n - \mu_x}{\sigma_x / \sqrt{n}} \leq x \right)$

and Φ is CDF of $Z \sim N(0,1)$

Today: $c < 0.71$ lower bound: $\frac{3+\sqrt{10}}{6\sqrt{2\pi}} \approx 0.4097.$

CLT Binomial Approximation

For large n: $b(n,k,p) \overset{d}{\approx} N(\overset{\mu_x}{np}, \overset{\sigma_x^2}{n \cdot pq})$ if $q = 1-p.$

Rule of thumb: and $\begin{cases} (1) & np \geq 5 \\ (2) & nq \geq 5 \end{cases}$

continuity correction.

Note: $P(3 \leq X < 6) \approx P(2.5 \leq X \leq 5.5)$

$P(3 \leq X \leq 6) \approx P(2.5 \leq X \leq 6.5)$

Ex: $X \sim b(10, \frac{1}{2})$

$\therefore n = 10 \qquad p = \frac{1}{2} \qquad \therefore np = n(1-p) = 10 \cdot \frac{1}{2} = 5.$

$\therefore$ OK to use CLT.
(but NOT Poisson Law)

$$\therefore P(3 \le X < 6) \overset{\text{continuity correction}}{\approx} P(2.5 \le X \le 5.5)$$

$$\overset{\text{standardize}}{=} P\left(\frac{2.5-np}{\sqrt{npq}} \le \frac{X-np}{\sqrt{npq}} \le \frac{5.5-np}{\sqrt{npq}}\right)$$

$$= P\left(\frac{2.5-5}{\sqrt{10/4}} \le \frac{X-5}{\sqrt{10/4}} \le \frac{5.5-5}{\sqrt{10/4}}\right)$$

$$\overset{\text{CLT}}{\approx} P(-1.58 \le Z \le 0.3162)$$

for $Z \sim N(0,1)$

$$= \phi(0.316) - \phi(-1.581)$$

$$\approx 0.5684.$$

Exact: $P(3 \le X < 6) = 0.5683$ if $X \sim b(10, \frac{1}{2})$

$$\therefore |0.5684 - 0.5683| = 0.0001$$

Confidence Intervals

2-sided for parameter θ:

$$\boxed{\hat{\theta}_n \pm \text{M.O.E.}}$$

M.O.E. = Margin of Error

usually involves $\sigma_{\hat{\theta}_n} = \sqrt{V[\hat{\theta}_n]}$ "standard error"

or its estimate $S_{\hat{\theta}_n}$

$\hat{\theta}_n$ usually involves $\overline{X}_n$, S_n^2, or S_{XY}

(random sets $X: \Omega \to \mathcal{B}$).

Ex: $(1-\alpha)\%$ C.I. for μ_X if σ_X^2 known and

random sample and Large $n \geq 30$

$$\text{C.I.} = \overline{X}_n \pm \text{M.O.E.}$$

$$= \overline{X}_n \pm z_{\alpha/2} \frac{\sigma_X}{\sqrt{n}}$$

$\sqrt{V[\overline{X}_n]}$ standard error

unknown σ_X^2:

S_n estimates σ_X

$$\text{C.I.} = \overline{X}_n \pm t_{\alpha/2}(n-1) \frac{S_n}{\sqrt{n}}$$

wider C.I.

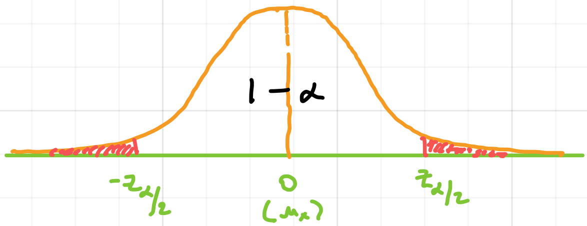
Thrm: $(1-\alpha)\,\%$ C.I. for μ_x if σ_x^2 known and large n:

(r.s.) $\Rightarrow$ IID, $\sigma_x^2 < \infty$

$$\boxed{\bar{X}_n \pm z_{\alpha/2} \frac{\sigma_x}{\sqrt{n}}}$$

memorize

Prf:



$$Z_n \triangleq \frac{X_n - \mu_x}{\sigma_x/\sqrt{n}}$$ since r.s.

$$\therefore \text{ large } n: \quad Z_n \underset{\text{CLT}}{\overset{d}{\approx}} Z \sim N(0,1)$$

$$\therefore \bar{X}_n \overset{d}{\approx} N\left(\mu_x, \frac{\sigma_x^2}{n}\right)$$

$$\therefore 1-\alpha \overset{\text{F.T.C.}}{=} P\left(-z_{\alpha/2} \le Z \le z_{\alpha/2}\right)$$

$$\overset{\text{CLT}}{\approx} P\left(-z_{\alpha/2} \le \frac{\bar{X}_n - \mu_x}{\sigma_x/\sqrt{n}} \le z_{\alpha/2}\right)$$ since n "large"

$$= P\left(\left\{\omega \in \Omega : -z_{\alpha/2} \le \frac{\bar{X}_n(\omega) - \mu_x}{\sigma_x/\sqrt{n}} \le z_{\alpha/2}\right\}\right)$$

$$= P\left(-z_{\alpha/2}\frac{\sigma_x}{\sqrt{n}} \le \bar{X}_n - \mu_x \le z_{\alpha/2}\cdot\frac{\sigma_x}{\sqrt{n}}\right)$$

$$= P\left(-\bar{X}_n - z_{\alpha/2}\frac{\sigma_x}{\sqrt{n}} \le -\mu_x \le -\bar{X}_n + z_{\alpha/2}\frac{\sigma_x}{\sqrt{n}}\right)$$

$$= P\left(\bar{X}_n + z_{\alpha/2}\frac{\sigma_x}{\sqrt{n}} \ge \mu_x \ge \bar{X}_n - z_{\alpha/2}\frac{\sigma_x}{\sqrt{n}}\right)$$

$$= P\left(\bar{X}_n - z_{\alpha/2}\frac{\sigma_x}{\sqrt{n}} \le \mu_x \le \bar{X}_n + z_{\alpha/2}\frac{\sigma_x}{\sqrt{n}}\right)$$

$\therefore \bar{X}_n \pm z_{\alpha/2}\frac{\sigma_x}{\sqrt{n}}$ is the $(1-\alpha)\,\%$ CI for μ_x QED.

Picking Sample Size n

Want: interval no wider than $\overline{X}_n \pm \varepsilon$ in above case

ε = Maximal error tolerance = M.O.E. = $z_{\alpha/2} \frac{\sigma_x}{\sqrt{n}}$

$\therefore \varepsilon^2 = z_{\alpha/2}^2 \cdot \frac{\sigma_x^2}{n}$.

$$\therefore \boxed{n = \frac{z_{\alpha/2}^2 \, \sigma_x^2}{\varepsilon^2}}$$ memorize

Proportions: Estimate with CLT:

$X \sim b(n,p)$ ← success/failure structure

proportion $\hat{p} \triangleq \frac{X}{n}$ ← # success, ← # trials

(MLE of p since $X \sim b(n,p)$)

① $E[\hat{p}] = \frac{E[X]}{n} \overset{X\sim b}{=} \frac{\cancel{n} \cdot p}{\cancel{n}} = p$ $\forall n$ $\therefore$ Unbiased

② $V[\hat{p}] = \frac{V[X]}{n^2} = \frac{\cancel{n} \cdot p \cdot q}{n^{\cancel{2}}} = \frac{p \cdot q}{n}$ ↓ 0 as $n \uparrow \infty$

$\therefore \hat{p}_n \xrightarrow{P} p$ by m.p.d. $\therefore \hat{p}_n$ is consistent for p.

$\therefore$ For large n: CLT $\hat{p}_n \overset{d}{\approx} N\left(p, \frac{p(1-p)}{n}\right)$

$\therefore (1-\alpha)\%$ C.I. for p: $\boxed{\hat{p} \pm z_{\alpha/2} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}}$

Ex: Polls. "$\pm 3\%$ M.O.E." in the media

implied $\varepsilon = 0.03$ at 95% C.I. $(\alpha = 0.05)$

since $p \in (0,1)$

Note: $p(1-p) \leq \frac{1}{4}$ $\therefore p = \frac{1}{2}$ maximizes $f(p) = p - p^2$

$\therefore V[\hat{p}] = \frac{p \cdot q}{n} \leq \frac{1}{4n}$

$\therefore$ Conservative estimate: $\sigma_{\hat{p}} \leq \frac{1}{2\sqrt{n}}$

$\therefore \varepsilon = \text{M.O.E.} = \frac{z_{\alpha/2}}{2\sqrt{n}}$

$\therefore \boxed{n = z_{\alpha/2}^2 / 4\varepsilon}$ pollster's sample size

$\therefore$ For 95% confidence Need $n \geq 1068$ samples

For 99% confidence $n \geq 1849$ samples

"Bootstrap" C.I.

for μ_x (or σ_x or σ_x^2) → Bradley-Efron (1979)

Need only iid random samples $X_1, X_2, \ldots, X_n$ – so long as they are "statistically representitive"

∴ "non-parametric" (no pdf. assumptions.)
↳ computer intensive

Ex: Find C.I. for μ_x given 100 r.s. $X_1, \ldots, X_{100}$

① Form 1000 "resamples" of size 100 each
– sample w/ replacement
↳ to create "synthetic data"
$R_1, R_2, \ldots, R_{1000}$

② Compute the sample mean for each resample R_k.

$$\begin{array}{cccc} R_1, & R_2, & \ldots, & R_{1000} \\ \downarrow & \downarrow & & \downarrow \\ \bar{X}_{R_1}, & \bar{X}_{R_2}, & \ldots, & \bar{X}_{R_{1000}} \end{array}$$

or: $\left(S^2_{R_1}, \quad S^2_{R_2}, \quad \ldots, \quad S^2_{R_{1000}}\right)$

③ Rank-order the 1000 samples

$$\bar{X}_{(1)} \le \bar{X}_{(2)} \le \cdots \le \bar{X}_{(25)} \le \cdots \le \bar{X}_{(975)} \le \cdots \le \bar{X}_{(1000)}$$

④ Pick percentiles 2.5 and 97.5 (for 95% CI):

CI: $\left(\bar{X}_{(25)}, \bar{X}_{(975)}\right)$ →
- consistent if "regularity" holds
- do same for S^2 or S, etc...

Thrm: If X and Y independent and $Z = X + Y$:

$$\boxed{f_Z = f_X * f_Y}$$

"splat" of convolution.

Prf: SIT technique (continuous case)

$$F_Z(z) \overset{\text{CDF}}{=} P[Z \le z] \overset{\text{Hyp}}{=} P[X + Y \le z]$$

$$\overset{\text{total prob}}{=} \int_{x=-\infty}^{x=\infty} P[X+Y \le z \mid X = x] \cdot f_X(x)\, dx$$

$$\overset{\text{sub}}{=} \int_{x=-\infty}^{x=\infty} P[x + Y \le z \mid X = x] \cdot f_X(x)\, dx.$$

$$\overset{\text{ind}}{=} \int_{x=-\infty}^{x=\infty} P[x + Y \le z]\, f_X(x)\, dx$$

$$= \int_{x=-\infty}^{x=\infty} P[Y \le z - x] \cdot f_X(x)\, dx.$$

$$\therefore f_Z(z) = \frac{d}{dz} F_Z(z)$$

$$= \frac{d}{dz} \int_{x=-\infty}^{x=\infty} F_Y(z-x) \cdot f_X(x)\, dx.$$

$$= \int_{x=-\infty}^{x=\infty} \frac{d}{dz} F_Y(z-x)\, f_X(x)\, dx \quad \text{(by Leibniz's Rule)}$$

$$= \int_{x=-\infty}^{x=\infty} f_Y(z-x) \cdot f_X(x)\, dx$$

$$= (f_Y * f_X)(z)$$

QED

Defn: Moment Generating Function (MGF)

$$M_x(s) = E_x[e^{sX}] \quad \text{for real } s \text{ where } E_x \text{ exists: } -c < s < c$$

(uniform convergence in discrete case)

$s = i\omega$ $(i = \sqrt{-1})$ gives:

Defn: Characteristic function (Levy, 1920's)

$$\phi_x(\omega) = E_x[e^{i\omega X}]$$

Note $\phi_x(\omega)$ always exists: $|\phi_x(\omega)| \leq E_x[|e^{i\omega X}|] = \phi_x(0) = 1$ $\forall \omega \in \mathbb{R}$

Thrm:

- $E_x[X^k] = \frac{d^k}{ds^k} M_x(s)\Big|_{s=0}$ "moment generating"
- $i^k \cdot E_x[X^k] = \frac{d^k}{d\omega^k} \phi_x(\omega)\Big|_{\omega=0}$

$\therefore \mu_x = M_x'(0)$ $\qquad \sigma_x^2 = M_x''(0) - (M_x'(0))^2$.

Prf: $k=1$ (continuous case when M_x exists)

$$\frac{d}{ds} M_x(s)\Big|_{s=0} = \frac{d}{ds} \int_{x=-\infty}^{x=\infty} e^{sx} f_x(x)\, dx \Big|_{s=0}$$

$$= \int_{x=-\infty}^{x=\infty} \frac{d}{ds} e^{sx}\Big|_{s=0} f_x(x)\, dx$$

$$= \int_{x=-\infty}^{x=\infty} x e^{sx}\Big|_{s=0} f_x(x)\, dx = \int_{x=-\infty}^{x=\infty} x \cdot e^{0} f_x(x)\, dx$$

$$= \int_{x=-\infty}^{x=\infty} x \cdot f_x(x)\, dx = E_x[X]$$

Repeat k times for $E_x[X^k]$. Put $s = i\omega$ for ϕ_x QED

Ex: $X \sim b(n,p)$ and $Y \sim P(\lambda)$

$$\therefore \mathcal{M}_X(s) = E_X[e^{sX}] \stackrel{b}{=} \sum_{k=0}^{n} e^{sk} \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} (pe^s)^k (1-p)^{n-k}$$

$$\stackrel{BT}{=} \boxed{(1-p+pe^s)^n}$$

$$\mathcal{M}_Y(s) = E_Y[e^{sY}] = \sum_{k=0}^{\infty} e^{sk} \frac{e^{-\lambda}\lambda^k}{k!}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^s)^k}{k!}$$

$$= \boxed{e^{\lambda(e^s-1)}}$$

☆ **Thrm:** (Levy's Continuity Theorem) (use to prove CLT)

$X_n \xrightarrow{d} X$ at points of continuity

iff $\phi_{X_n} \xrightarrow{e} \phi_X$ (and ϕ_X is continuous)

Ex: Poisson Law $b \xrightarrow{d} p$ if $\lambda = n \cdot p$

$$\lim_{n\to\infty} \mathcal{M}_X(s) = \lim_{n\to\infty} (1-p+pe^s)^n \quad \text{if } X \sim b(n,p)$$

$$\stackrel{\lambda=np}{=} \lim_{n\to\infty} \left(1 - \frac{\lambda}{n} + \frac{\lambda e^s}{n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{\lambda(e^s-1)}{n}\right)^n$$

$$= e^{\lambda(e^s-1)} = \mathcal{M}_Y(s) \quad \text{if } Y \sim P(\lambda)$$

$\therefore b \xrightarrow{d} p$ by Levy's Theorem

Thrm: If $X \sim N(\mu_x, \sigma_x^2)$

memorize

1-D
$$\begin{cases} \mathcal{M}_x(s) = e^{s\mu + s^2\sigma^2/2} \\ \phi_x(w) = e^{iw\mu - w^2\sigma^2/2} \end{cases}$$

If $\underset{n\times 1}{X} \sim N(\underset{n\times 1}{\mu_x}, \underset{n\times n}{K_{xx}})$ positive semi-definite

n-D
$$\begin{cases} \mathcal{M}_x(s) = \exp\left(s^T\mu_x + \frac{s^T K_{xx} s}{2} \right) \\ \phi_x(w) = \exp\left(iw\mu_x - \frac{w^T K_{xx} w}{2} \right) \end{cases}$$

Note: K_{xx}^{-1} need not exist (unlike pdf f_x)
eigenvalues $\lambda = 0$ OK.

Prf: (1-D case)

$X \sim N(\mu, \sigma^2)$

$$\mathcal{M}_x(s) = E_x[e^{sX}] = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{x=-\infty}^{x=\infty} e^{sx} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

put $z = \frac{x-\mu}{\sigma}$ $\therefore x = \sigma z + \mu$ $dx = \sigma\, dz$
$(x = \pm\infty \longrightarrow z = \pm\infty)$

$$\overset{z = \frac{x-\mu}{\sigma}}{=} \frac{1}{\sqrt{2\pi}\,\cancel{\sigma}} \int_{z=-\infty}^{z=\infty} e^{s\sigma z + s\mu} e^{-z^2/2}\, \cancel{\sigma} \cdot dz$$

$$= \frac{e^{s\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2zs\sigma)} dz$$

$$\overset{\text{complete the square}}{=} \frac{e^{s\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}[(z-s\sigma)^2 - s^2\sigma^2]} dz.$$

since $(z-s\sigma)^2 - s^2\sigma^2 = z^2 + s^2\sigma^2 - 2zs\sigma - s^2\sigma^2 = z^2 - 2zs\sigma$

$$= e^{s\mu + \frac{s^2\sigma^2}{2}} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-s\sigma)^2} dz \right.$$

put $u = z - s\sigma$ $\therefore du = dz$, $z = \pm\infty \longrightarrow u = \pm\infty$

$$\overset{u=z-s\sigma}{=} e^{s\mu + s^2\sigma^2/2} \underbrace{\left(\frac{1}{\sqrt{2\pi}} \int_{u=-\infty}^{u=\infty} e^{-u^2/2}\, du \right)}_{=1 \text{ since pdf}}$$

$$= e^{s\mu + s^2\sigma^2/2}$$

QED.

Proposition: If X and Y independent and $Z = X + Y$

$$\begin{cases} \mathcal{M}_Z = \mathcal{M}_X \mathcal{M}_Y \\ \phi_Z = \phi_X \phi_Y \end{cases}$$

n-d: $\mathcal{M}_{\sum_{k=1}^{n} X_k} = \prod_{k=1}^{n} X_k$

Prf: $\mathcal{M}_Z(s) = E_{XY}\left[e^{sX+sY}\right] = E_{XY}\left[e^{sX} e^{sY}\right]$

$$\overset{\text{ind}}{=} E_{XY}\left[e^{sX}\right] \cdot E_{XY}\left[e^{sY}\right] = E_X\left[e^{sX}\right] \cdot E_Y\left[e^{sY}\right].$$

$$= \mathcal{M}_X(s) \cdot \mathcal{M}_Y(s)$$

QED.

Ex: Cauchy $X \sim C(m, d)$: $f_X(x) = \dfrac{1}{\pi d\left(1 + \left(\frac{x-m}{d}\right)^2\right)}$

$\mathcal{M}_X$ does not exist, But ϕ_X does

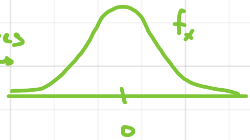
$$\boxed{\phi_X(\omega) = e^{i\omega m - d|\omega|}}$$

from Cauchy's Residue Theorem:
$Z \sim C(0,1)$: $\phi_Z(\omega) = e^{-|\omega|}$

General Symmetric Alpha Stable (SαS) (Paul Levy, 1923)

$$\phi_x(\omega) = e^{-d|\omega|^\alpha}$$

bell curves



for tail-thickness $\alpha \in (0, 2]$

No closed form pdf except

$\alpha = \frac{1}{2}$ (Levy) asymmetric
$\alpha = 1$ (Cauchy)
$\alpha = 2$ (Gaussian)

$\alpha \downarrow \longleftrightarrow$ bell-curve tail thickness $\uparrow$

$E_x[|X|^f]$ exists if $f < \alpha$

(lower-order Fractional moments)

Higher-order moments do not exist if $\alpha < 2$

$E[|X|^f] = \infty$ if $f \geq \alpha$

General Stable Distributions

$$\phi_x(\omega) = \begin{cases} \exp\left[-\gamma^\alpha|\omega|\left[1+\beta \tan\frac{\alpha\pi}{2}\cdot \text{sign}(\omega)\left(|\gamma\omega|^{1-\alpha}-1\right)\right] + im\omega\right] & \text{if } \alpha \neq 1 \\ \exp\left[-\gamma|\omega|\left[1+\beta\frac{2}{\pi}(\text{sign}\,\omega)(\text{Log}|\omega|)\right] + im\omega\right] & \text{if } \alpha = 1 \end{cases}$$

- tail-thickness $0 < \alpha \leq 2$
- dispersion $\gamma > 0$
- location $m \in \mathbb{R}$
- skew $-1 \leq \beta \leq 1$ ($\beta = 0$ symmetric pdf)

$\star$ Inversion (Transform) Theorem

If CDF F_X is absolutely continuous (so pdf f_x exists):

$$\boxed{f_x(w) = \frac{1}{2\pi}\int_{w=-\infty}^{w=\infty} \phi_x(w)\cdot e^{-iwx}\, dw}$$

Ex: $X \sim C(m, d)$

$$\therefore f_x(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iwm - d|w|}\, e^{-ixw}\, dw$$

$$= \frac{1}{2\pi}\left[\int_0^{\infty} e^{iwm - dw}\, e^{-iwx}\, dw + \int_{-\infty}^{0} e^{iwm + dw}\, e^{-iwx}\, dw\right]$$

since $|w| = \begin{cases} w & \text{if } w \geq 0 \\ -w & \text{if } w < 0 \end{cases}$

$$= \frac{1}{2\pi}\left[\int_0^{\infty} e^{-w(d+ix-im)}\, dw + \int_{-\infty}^{0} e^{w(d-ix+im)}\, dw\right]$$

$u = -w(d+ix-im)$
$\therefore du = -dw(d+ix-im)$
$\begin{bmatrix} w=0 \longrightarrow u=0 \\ w=\infty \longrightarrow u=-\infty \end{bmatrix}$

$v = w(d-ix+im)$
$dv = dw(d-ix+im)$
$\begin{bmatrix} w=0 \longrightarrow v=0 \\ w=-\infty \longrightarrow v=-\infty \end{bmatrix}$

$$= \frac{1}{2\pi}\left[-\frac{1}{d+ix-im}\int_{u=0}^{u=-\infty} e^u\, du + \frac{1}{d-ix+im}\int_{v=-\infty}^{v=0} e^v\, dv\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^u}{d+ix-im}\Bigg|_{-\infty}^{0} + \frac{e^v}{d-ix+im}\Bigg|_{-\infty}^{0}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{d+ix-im} + \frac{1}{d-ix+im}\right]$$

$$= \frac{1}{\cancel{2}\pi}\left[\frac{\cancel{2}d}{d^2+x^2+m^2-2mx}\right] \quad \text{after rationalizing}$$

$$= \frac{1}{\pi d}\cdot\frac{1}{\left(1+\left(\frac{x-m}{d}\right)^2\right)}$$

$$= f_x(x) \longleftrightarrow X \sim C(m, d)$$

Thrm: If independent $X_1, \dots, X_n$ and $X_k \sim N(\mu_k, \sigma_k^2)$ then

$$\sum_{k=1}^n a_k X_k \sim N\left(\sum_{k=1}^n a_k \mu_k, \sum_{k=1}^n a_k^2 \sigma_k^2\right)$$

Prf:

$$\phi_{\sum_{k=1}^n a_k X_k}(\omega) = E\left[e^{i\omega \sum_{k=1}^n a_k X_k}\right]$$

$$= E\left[\prod_{k=1}^n e^{i\omega a_k X_k}\right]$$

$$\overset{\text{ind}}{=} \prod_{k=1}^n E\left[e^{i(\omega a_k) X_k}\right]$$

$$= \prod_{k=1}^n \phi_{X_k}(\omega a_k)$$

$$= \prod_{k=1}^n e^{i\omega a_k \mu_k - \frac{\omega^2}{2} a_k^2 \cdot \sigma_k^2}$$

$$= e^{i\omega\left(\sum_{k=1}^n a_k \mu_k\right) - \frac{\omega^2}{2}\left(\sum_{k=1}^n a_k^2 \sigma_k^2\right)}$$

$$\longleftrightarrow \sum_{k=1}^n a_k X_k \sim N\left(\sum_{k=1}^n a_k \mu_k, \sum_{k=1}^n a_k^2 \sigma_k^2\right)$$

QED.

Q: How does $\bar{X}_n$ differ if (iid) $X_k \sim N(\mu, \sigma^2)$ vs. $X_k \sim C(m, d)$

Thrm: If iid $X_1, \dots, X_n \sim N(\mu, \sigma^2)$: $\bar{X}_n \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

(exact, not CLT approximation)

Prf:

$$\phi_{\bar{X}_n}(\omega) = E\left[e^{i\omega \bar{X}_n}\right] = E\left[e^{i\omega \frac{1}{n}\sum_{k=1}^n X_k}\right]$$

$$= E\left[e^{\sum_{k=1}^n i \frac{\omega}{n} X_k}\right] = E\left[\prod_{k=1}^n e^{i\frac{\omega}{n} X_k}\right]$$

$$\overset{\text{ind}}{=} \prod_{k=1}^n E\left[e^{i\left(\frac{\omega}{n}\right) X_k}\right] = \prod_{k=1}^n \phi_{X_k}\left(\frac{\omega}{n}\right)$$

$$\overset{X \sim N(\mu, \sigma^2)}{=} \prod_{k=1}^n e^{i\frac{\omega}{n}\mu_k - \frac{\omega^2}{n^2}\frac{\sigma_k^2}{2}}$$

$$\overset{\text{Id.}}{=} \left(e^{i\left(\frac{\omega}{n}\right)\mu - \frac{\omega^2}{2}\left(\sigma^2/n^2\right)}\right)^n$$

$$= e^{i\omega\mu - \frac{\omega^2}{2}\frac{\sigma^2}{n}} \longleftrightarrow \bar{X}_n \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

QED

★

Thrm If iid $X_1, \dots, X_n \sim C(m,d)$: $\bar{X}_n \sim C(m,d)$

$\therefore$ no benefit to averaging!

Prf: $\phi_{\bar{X}_n}(\omega) = \prod_{k=1}^{n} E\left[e^{i\left(\frac{\omega}{n}\right)X_k}\right]$ (as in previous proof)

$$= \prod_{k=1}^{n} \phi_{X_k}\left(\frac{\omega}{n}\right)$$

$$\overset{X_k \sim C(m,d)}{=} \prod_{k=1}^{n} e^{i\left(\frac{\omega}{n}\right)m - d\left|\frac{\omega}{n}\right|}$$

$$= e^{i\sum_{k=1}^{n}\left(\left(\frac{\omega}{n}\right)m - d\left|\frac{\omega}{n}\right|\right)}$$

$$\overset{\text{Id.}}{=} e^{i\left(\cancel{n}\cdot\frac{\omega}{\cancel{n}}\cdot m - \cancel{n}\cdot d\,\frac{|\omega|}{\cancel{n}}\right)}$$

$$= e^{i\omega m - d|\omega|} \longleftrightarrow \bar{X}_n \sim C(m,d)$$

QED

$V[\bar{X}_n] = \frac{\sigma_x^2}{n}$ requires $\sigma_x^2 < \infty$

Note: Francis Edgeworth showed in 1883 that $\sigma_x^2 < \infty$ is <u>not</u> sufficient, even when $n=2$.

Central Limit Theorem ☆☆☆

Thrm: If iid $X_1, X_2, \ldots$ and $\sigma^2 < \infty$ $(\therefore |\mu_x| < \infty)$ then

$$Z_n \xrightarrow{d} Z \sim N(0,1)$$

where $Z_n = \text{STD}(\overline{X}_n) = \dfrac{\overline{X}_n - \mu}{\sigma/\sqrt{n}}$

$$= \text{STD}\left(\sum_{k=1}^{n} X_k\right) = \frac{\sum^{n} X_k - n\cdot\mu}{\sqrt{n}\cdot\sigma}$$

Prf: Strategy: show $M_{Z_n} \xrightarrow{e} M_Z(s) = e^{s^2/2}$ and then Levy's theorem.

$$M_{Z_n}(s) = E\left[e^{sZ_n}\right] = E\left[e^{s\left(\frac{\sum^{n} X_k - n\cdot\mu}{\sqrt{n}\sigma}\right)}\right]$$

$$= E\left[e^{\sum_{k=1}^{n}\frac{s}{\sqrt{n}\sigma}(X_k-\mu)}\right]$$

$$= E\left[\prod_{k=1}^{n} e^{\frac{s}{\sqrt{n}\sigma}(X_k-\mu)}\right] \overset{\text{ind}}{=} \prod_{k=1}^{n} E\left[e^{\frac{s}{\sqrt{n}\sigma}(X_k-\mu)}\right]$$

$$= E^n\left[e^{\frac{s(X-\mu)}{\sqrt{n}\sigma}}\right]$$

$$= \left(E\left[1 + \frac{s(X-\mu)}{\sqrt{n}\,\sigma} + \frac{s^2(X-\mu)^2}{2n\sigma^2} + \text{H.O.T.}\right]\right)^n$$

$$= \left(1 + 0 + \frac{s^2\sigma^2}{2n\sigma^2} + \text{H.O.T.}'\right)^n$$

$$= \left(1 + \frac{s^2/2}{n} + \text{H.O.T.}'\right)^n \xrightarrow{e} e^{s^2/2}$$

since $\lim_{n\to\infty}\left(1+\frac{c_n}{n}\right)^n = e^c$ if $\lim_{n\to\infty} c_n = c$

$$= M_Z(s) \longleftrightarrow Z \sim N(0,1)$$

$\therefore Z_n \xrightarrow{d} Z$ by Levy's continuity theorem, since $e^{s^2/2}$ is continuous

QED

$\therefore$ "Delta Method" for continuous (differentiable) g (not involving n)

If $\sqrt{n}(X_n - \mu) \xrightarrow{d} N(0, \sigma^2)$

then $\sqrt{n}(g(X_n) - g(\mu)) \xrightarrow{d} N(0, \sigma^2 [g'(\mu)]^2)$

follows from mean-value theorem and continuity of Plim.

"Lindeberg condition" yields the CLT when only independence hold (iid need not)

Thrm: (Lindeberg-Feller CLT)

If
- $X_1, X_2, \ldots$ are independent and $\sigma_k^2 < \infty$
- $\lim_{n\to\infty} \bar{\sigma}_n^2 = \bar{\sigma}^2 < \infty$ for $\bar{\sigma}_n^2 = \frac{1}{n}(\sigma_1^2 + \cdots + \sigma_n^2)$
- $\lim_{n\to\infty} \frac{\max_k \sigma_k}{n \cdot \bar{\sigma}_n} = 0$ (i.e. no term dominates the average variance)

Then: $\sqrt{n}(\bar{X}_n - \bar{\mu}_n) \xrightarrow{d} N(0, \bar{\sigma}^2)$

for $\bar{\mu}_n = \frac{1}{n}(\mu_1 + \cdots + \mu_n)$

Thrm: (Generalized CLT), CLT holds $\longleftrightarrow$ stable r.v.

iid $X_1, X_2, \ldots$ then $\exists\, a_n > 0$ and $b_n \in \mathbb{R}$ and non-degenerate

r.v. Z : $a_n \sum_{k=1}^{n} X_k - b_k \xrightarrow{d} Z$ iff Z is α-stable for some $\alpha \in (0,2]$

$\therefore$ Classical CLT if $Z \sim N(0,1)$ $(\alpha = 2)$ and $a_n = \frac{1}{\sqrt{n}\,\sigma}$ and $b_n = \sqrt{n}\, \frac{\mu}{\sigma}$

Thrm: (SαS case of CLT-like averaging) If iid

$X_1, X_2, \ldots, X_n \sim \phi_x(\omega) = e^{-c_\alpha |\omega|^\alpha}$ and $0 < \alpha \leq 2$ then

$$\frac{1}{n^{1/\alpha}} \sum_{k=1}^{n} X_k \sim \phi_x(\omega) = e^{-c_\alpha |\omega|^\alpha}$$

Already saw this for $\alpha = 2$: $\bar{X}_n \sim N(\mu, \frac{\sigma^2}{n})$ and

for $\alpha = 1$: $\bar{X}_n \sim C(m, d)$

Extreme Value "CLT"

- extreme value. $P(X > 0)$

  $M_n \triangleq \max(X_1, \ldots, X_n)$

  dual result for min since $\min(x_1, \ldots, x_n) = -\max(x_1, \ldots, x_n)$

- M_n degenerates as $n \longrightarrow \infty$

- need to find a_n and b_n that <u>normalize</u> M_n

- turns out only 3 limiting distribution

Thrm: (Extreme Value Theorem, Fisher-Tippett Theorem) If

(1) $X_1, X_2, \ldots$ are iid

(2) $\exists\, a_n > 0$ and $b_n \in \mathbb{R}$: $\lim_{n \to \infty} P\left(\frac{M_n - b_n}{a_n} \leq x\right) = F_x(x)$

and F is not a degenerate CDF for $M_n = \max(X_1, \ldots, X_n)$

then :

F must be one of 3 "attractor" CDFs :

$$F \begin{cases} \text{Gumbel CDF} \\ \text{or} \\ \text{Fréchet CDF} \\ \text{or} \\ \text{Weibull CDF} \end{cases}$$

Ex: $X_k \sim N(\mu, \sigma^2)$, $X_k \sim \exp(\theta)$, $X_k \sim \text{lognormal}(\mu, \sigma^2)$

$$\therefore\ F \sim \text{Gumbel} : \ F_X(x) = e^{-e^{\left(\frac{\mu - x}{\beta}\right)}}$$

Ex: $X_k \sim C(m, d)$, $X \sim S\alpha S$ for $\alpha << 2$, $X_k \sim t(n-1)$ student, $X_k \sim$ Pareto

$$\therefore\ F \sim \text{Fréchet} : \ F_X(x) = e^{-\left(\frac{x-\mu}{d}\right)^{-\alpha}} \text{ if } \alpha \in (0, \infty)$$

Ex: $X_k \sim U(a, b)$, $X_k \sim \text{Beta}(\alpha, \beta)$

$$\therefore\ F \sim \text{Weibull} : \ F_X(x) = 1 - e^{-(x/\beta)^k}$$

Ex: $X_k \sim \text{Poisson}(\lambda)$

$\therefore$ no attractor CDF (degenerate)

Ex: Pick $a_n = 1 > 0$ and $b_n = \ln n$ and $X_k \sim \exp(1)$ iid

$$\therefore\ P\left[\frac{M_n - b_n}{a_n} \le x\right] = P\left[M_n \le a_n x + b_n\right] \overset{\text{iid}}{=} P^n(X \le a_n x + b_n)$$

since $M_n = \max(X_1, \dots, X_n)$

$$\overset{X_k \sim \exp(1)}{=} \left(1 - e^{-a_n x - b_n}\right)^n \overset{a_n = 1,\ b_n = \ln n}{=} \left(1 - e^{-x - \ln n}\right)^n = \left(1 - e^{\ln\left(\frac{1}{n}\right)} e^{-x}\right)^n$$

$$= \left(1 - \frac{e^{-x}}{n}\right)^n \xrightarrow{d} e^{-e^{-x}} = F_X(x) \longleftrightarrow \text{Gumbel CDF.}$$

Gamma pdf

$X \sim \gamma(\alpha, \theta)$, $\therefore f_x(x) = \dfrac{x^{\alpha-1} e^{-x/\theta}}{\Gamma(\alpha)\, \theta^{\alpha}}$ if $x > 0$ ($\alpha > 0$ and $\theta > 0$)

for $\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x}\, dx$

$\therefore \Gamma(\alpha+1) = \alpha \cdot \Gamma(\alpha)$ from integration by parts

Thrm: Say $X \sim \gamma(\alpha, \theta)$

① $E[X^k] = \theta^k \dfrac{\Gamma(\alpha+k)}{\Gamma(\alpha)}$ if $k > 0$

moments

$\therefore E[X] = \alpha\theta \qquad V[X] = \alpha\theta^2$

② MGF : $M_X(s) = \dfrac{1}{(1-s\theta)^{\alpha}}$ if $s < \frac{1}{\theta}$

③ Additivity. Say $X_1, \dots, X_n$ are independent and

same θ

$X_k \sim \gamma(\alpha_k, \theta)$. Then $X = \sum_{k=1}^{n} X_k \sim \gamma\left(\sum_{k=1}^{n} \alpha_k, \theta\right)$.

$\therefore \sum_{k=1}^{r} X_k \sim \chi^2(r)$ if iid $X_k \sim \chi^2(1)$

since $X \sim \chi^2(r)$ if $X \sim \gamma(\frac{r}{2}, 2)$

Pf:

① $E[X^k] = \dfrac{1}{\Gamma(\alpha)\theta^{\alpha}} \int_0^{\infty} x^k x^{\alpha-1} e^{-x/\theta}\, d\theta$

$= \dfrac{1}{\Gamma(\alpha)\theta^{\alpha}} \int_0^{\infty} x^{(\alpha+k)-1} e^{-x/\theta}\, d\theta$

put $u = x/\theta$ $\therefore x = \theta u$ and $dx = \theta du$

$x = 0 \longrightarrow u = 0$ since $\theta > 0$

$x = \infty \longrightarrow u = \infty$

$$= \frac{1}{\Gamma(\alpha)\Theta^\alpha} \int_{u=0}^{u=\infty} \Theta^{(\alpha+k)-1} u^{(\alpha+k)-1} e^{-u} \Theta \, du.$$

$$= \frac{\Theta^{\cancel{\alpha}+k}}{\Gamma(\alpha)\cancel{\Theta^\alpha}} \underbrace{\int_0^\infty u^{(\alpha+k)-1} e^{-u} \, du}_{\Gamma(\alpha+k)}$$

$$= \Theta^k \cdot \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}$$

$$\therefore E[X^1] = \Theta^1 \cdot \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} = \Theta \cdot \frac{\alpha \cdot \Gamma(\alpha)}{\Gamma(\alpha)} = \alpha\Theta.$$

$$E[X^2] = \Theta^2 \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)} = \Theta^2 \cdot \frac{(\alpha+1)\alpha\cancel{\Gamma(\alpha)}}{\cancel{\Gamma(\alpha)}} = \Theta^2 \alpha(\alpha+1)$$

$$= \alpha^2\Theta^2 + \alpha\Theta^2$$

$$\therefore V[X] = E[X^2] - E^2[X] = \cancel{\alpha^2\Theta^2} + \alpha\Theta^2 - \cancel{\alpha^2\Theta^2}$$

$$= \alpha\Theta^2.$$

QED (1)

② $$M_X(s) = E[e^{sX}] = \frac{1}{\Gamma(\alpha)\Theta^\alpha} \int_0^\infty e^{sx} x^{\alpha-1} e^{-x/\Theta} \, d\Theta$$

$$= \frac{1}{\Gamma(\alpha)\Theta^\alpha} \int_0^\infty x^{\alpha-1} e^{-x\left(\frac{1}{\Theta}-s\right)} \, dx \qquad \text{if } s < \frac{1}{\Theta}$$

$$= \frac{1}{\Gamma(\alpha)\Theta^\alpha} \int_0^\infty x^{\alpha-1} e^{-\frac{x(1-s\Theta)}{\Theta}} \, dx$$

put $u = \frac{x(1-s\Theta)}{\Theta} \longleftrightarrow x = \frac{\Theta u}{1-s\Theta}$ and $dx = \frac{\Theta \, du}{1-s\Theta}$

$$= \frac{1}{\Gamma(\alpha)\Theta^\alpha} \int_{u=0}^{u=\infty} \frac{\Theta^{\alpha-1} u^{\alpha-1}}{(1-s\Theta)^{\alpha-1}} \frac{e^{-u}\,\Theta \, du}{1-s\Theta}$$

$$= \frac{\cancel{\Theta^\alpha}}{\cancel{\Gamma(\alpha)}\cancel{\Theta^\alpha}} \cdot \frac{\cancel{\Gamma(\alpha)}}{(1-s\Theta)^\alpha} = \frac{1}{(1-s\Theta)^\alpha} \qquad \text{if } s < \frac{1}{\Theta}$$

QED (2)

③ Say $X_k \sim \gamma(\alpha_k, \Theta)$ $\quad \therefore M_{X_k}(s) \overset{(2)}{=} \frac{1}{(1-s\Theta)^{\alpha_k}}$ if $s < \frac{1}{\Theta}$

$$\therefore M_X(s) = E[e^{sX}] = E\left[e^{s\sum_{k=1}^n X_k}\right] = E\left[e^{\sum_{k=1}^n sX_k}\right]$$

$$= E\left[\prod_{k=1}^{n} e^{sX_k}\right] \overset{\text{ind}}{=} \prod_{k=1}^{n} E\left[e^{sX_k}\right] = \prod_{k=1}^{n} m_{X_k}(s)$$

$$\overset{\gamma}{=} \prod_{k=1}^{n} (1-s\theta)^{-\alpha_k} = (1-s\theta)^{-\sum_{k=1}^{n}\alpha_k}$$

$$\longleftrightarrow \quad X \sim \gamma\left(\sum_{k=1}^{n}\alpha_k, \theta\right)$$

QED (3)

CLT and χ^2

Say $X \sim \chi^2(r)$ $\therefore X = \sum_{k=1}^{r} X_k$ with $\begin{cases} ① \text{ independent } X_1, \dots, X_n \\ ② \ X_k \sim \chi^2(1) \text{ all } k \end{cases}$

$\therefore \mu_{X_k} = 1$ and $\sigma^2_{X_k} = 2$ (since d.f. = 1)

CLT: iid $\sum_{k=1}^{r} X_k \overset{d}{\approx} N(r\mu, r\sigma^2)$ for "large" r

$\therefore X \overset{d}{\approx} N(r\mu, r\sigma^2)$ if $X \sim \chi^2(r)$ and large r

$= N(r \cdot 1, r \cdot 2)$

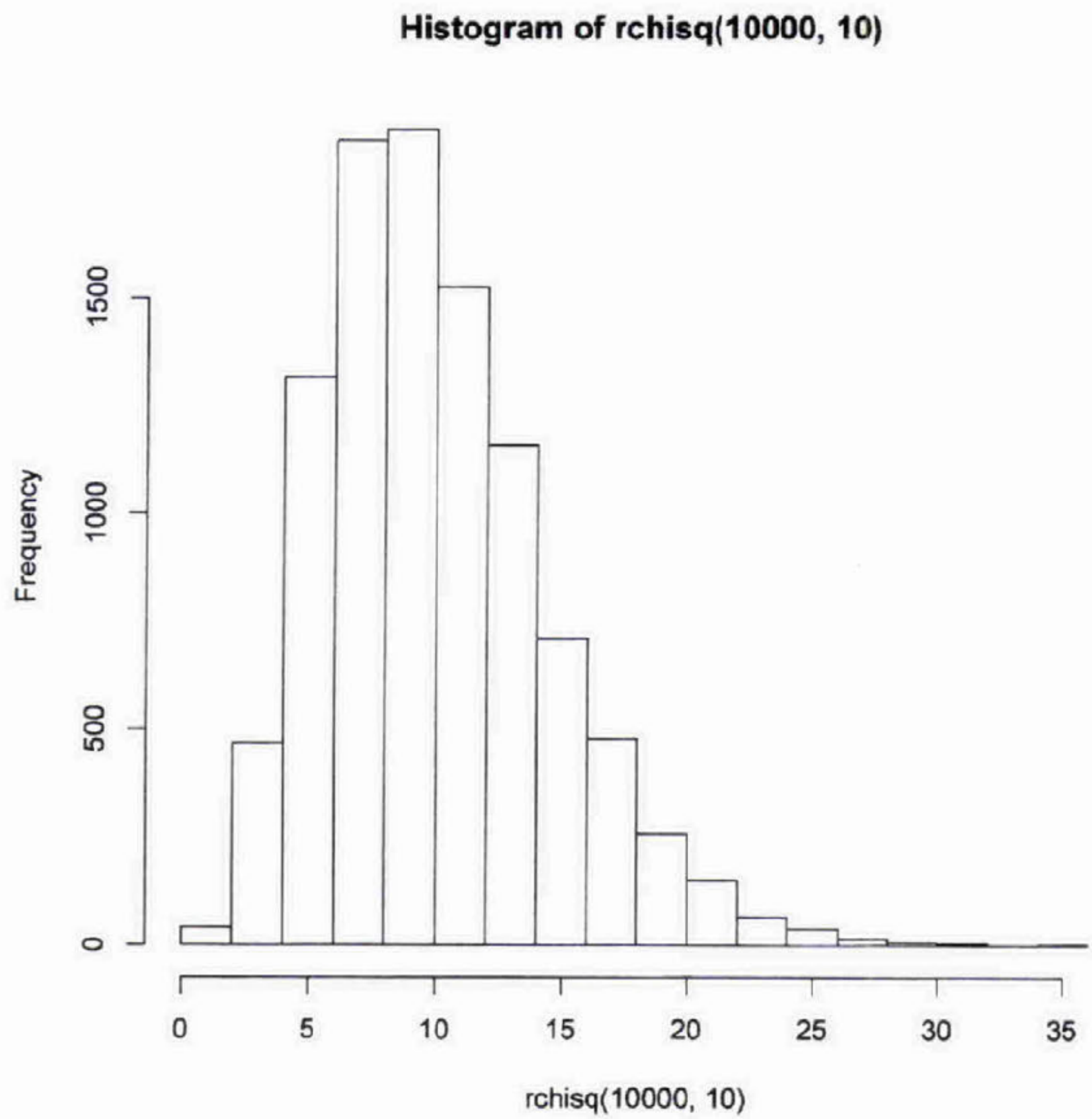
$\therefore X \overset{d}{\approx} N(r, 2r)$

Similarly: $\frac{1}{r}X = \frac{1}{r}\sum_{k=1}^{r} X_k = \overline{X}_r \overset{\text{CLT}}{\approx} N\left(1, \frac{2}{r}\right)$

Ex: (in R) 10,000 iid realizations of $X \sim \chi^2(r)$

Compare: hist(rchisq(10000, 10)) r=10

hist(rchisq(10000, 100)) r=100

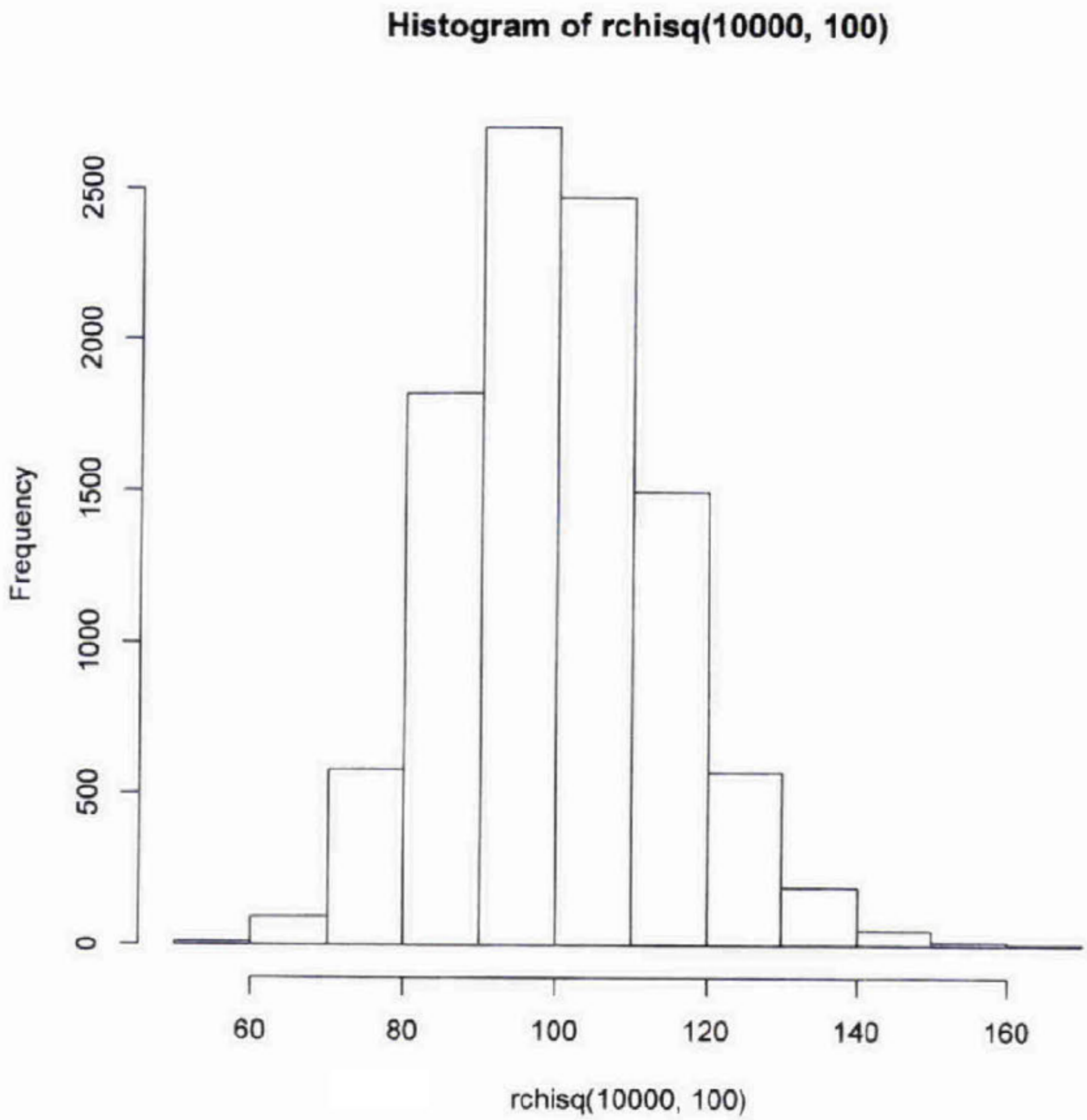


10,000 iid samples from $X \sim \chi^2(r)$

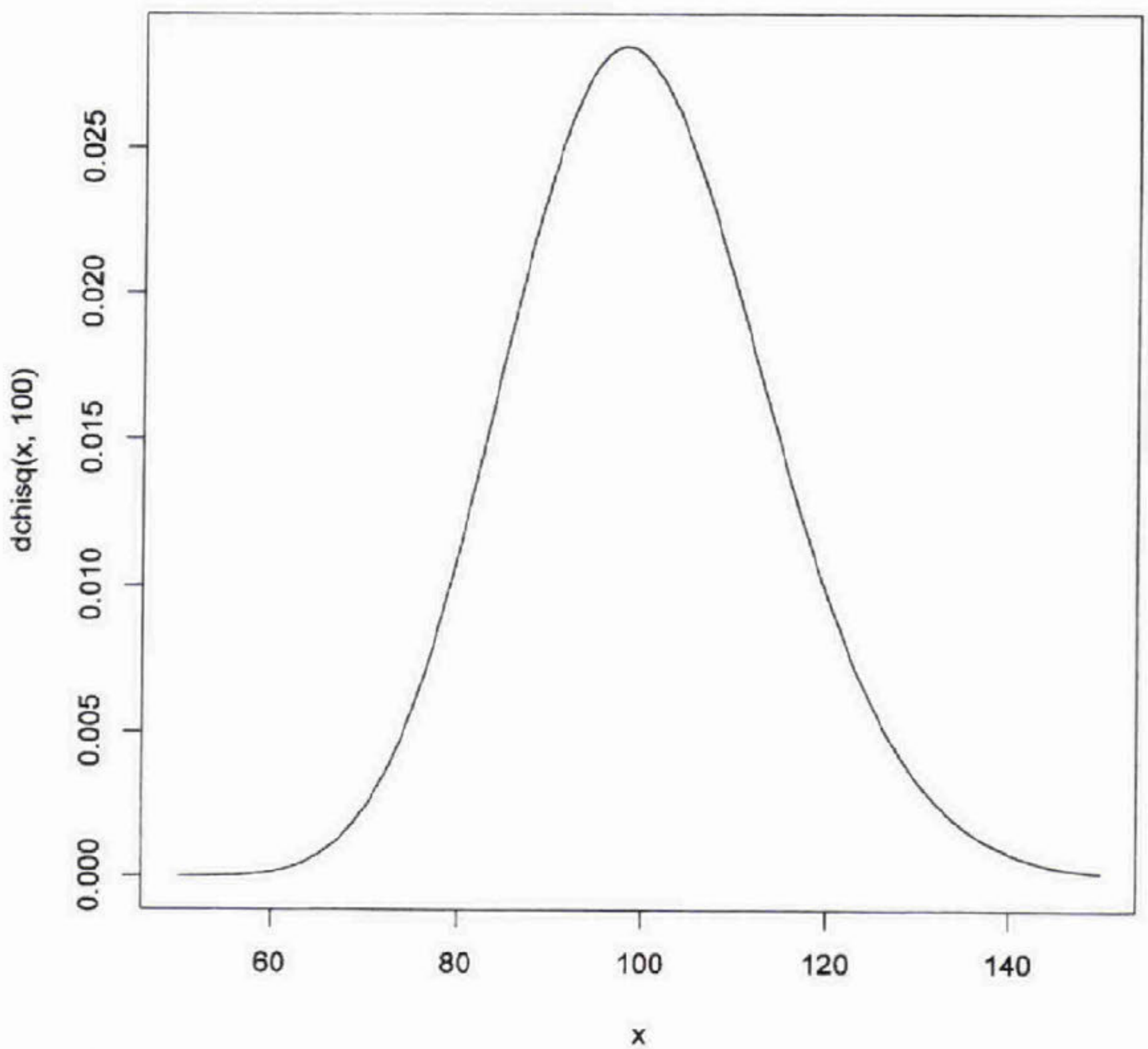
$r = 10$ d.f.

R command: hist (rchisq (10000, 10))

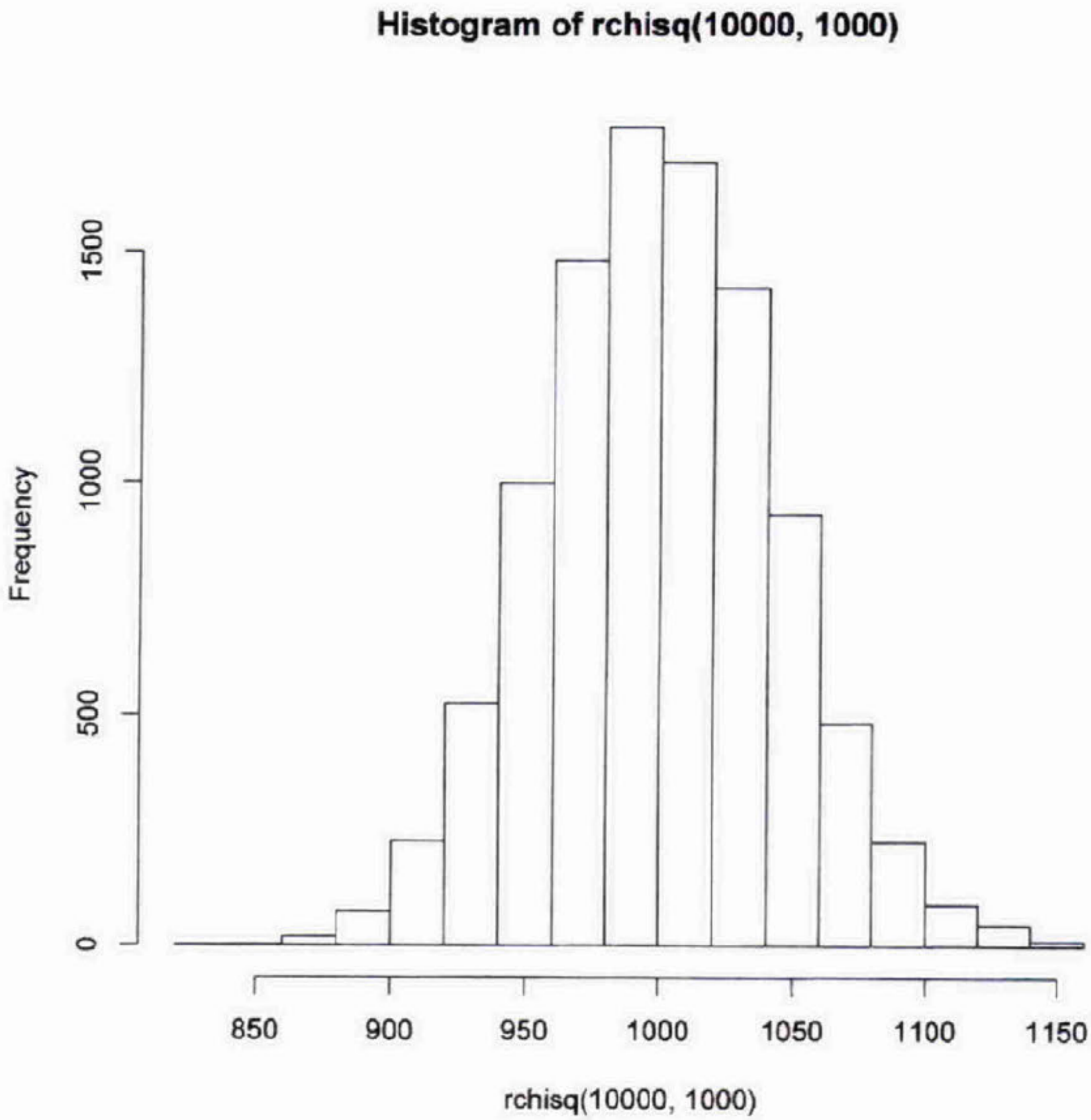
$n = \#$ random samples



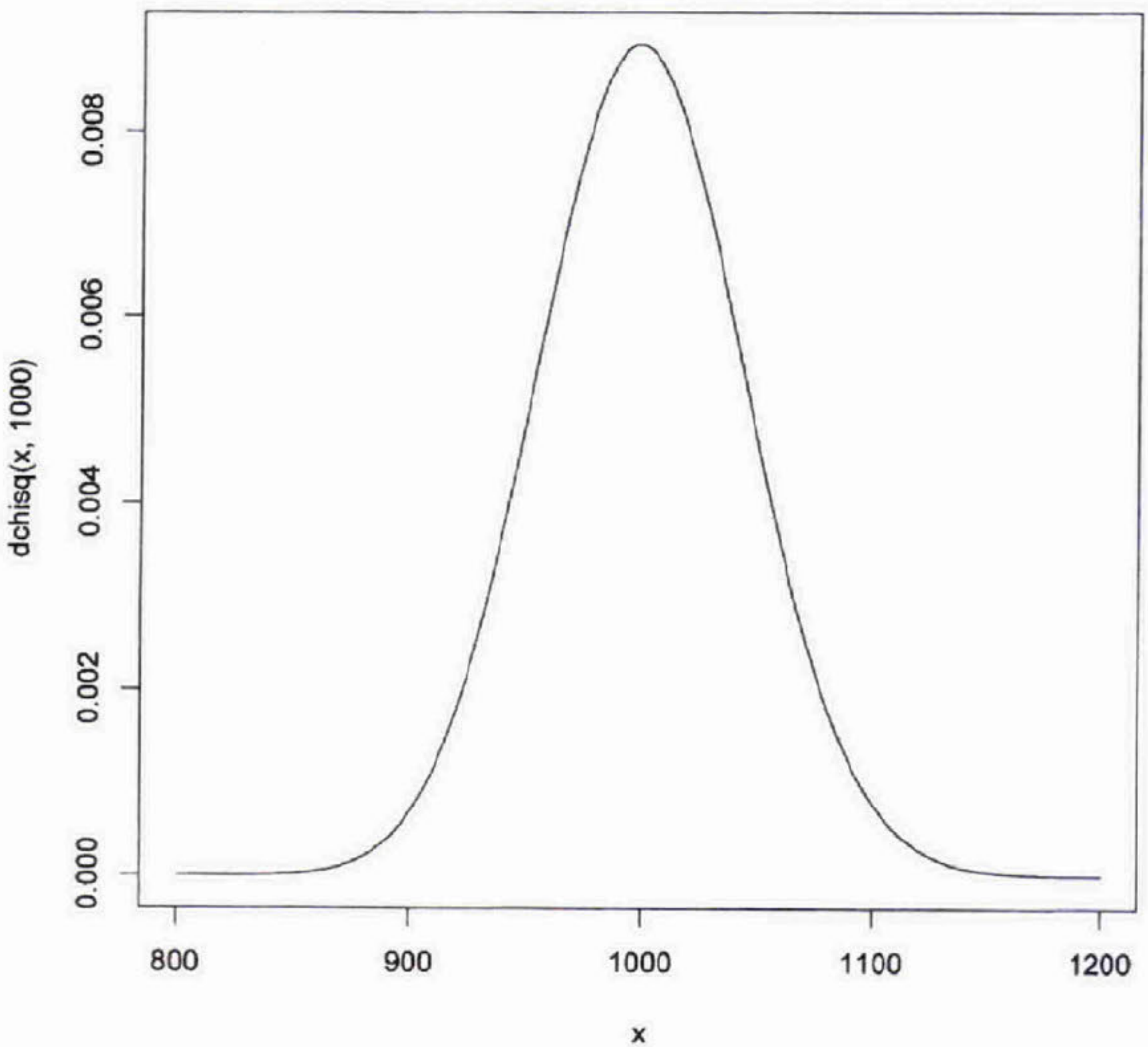
10,000 iid samples from $X \sim \chi^2(100)$



pdf of $X \sim \chi^2(100)$



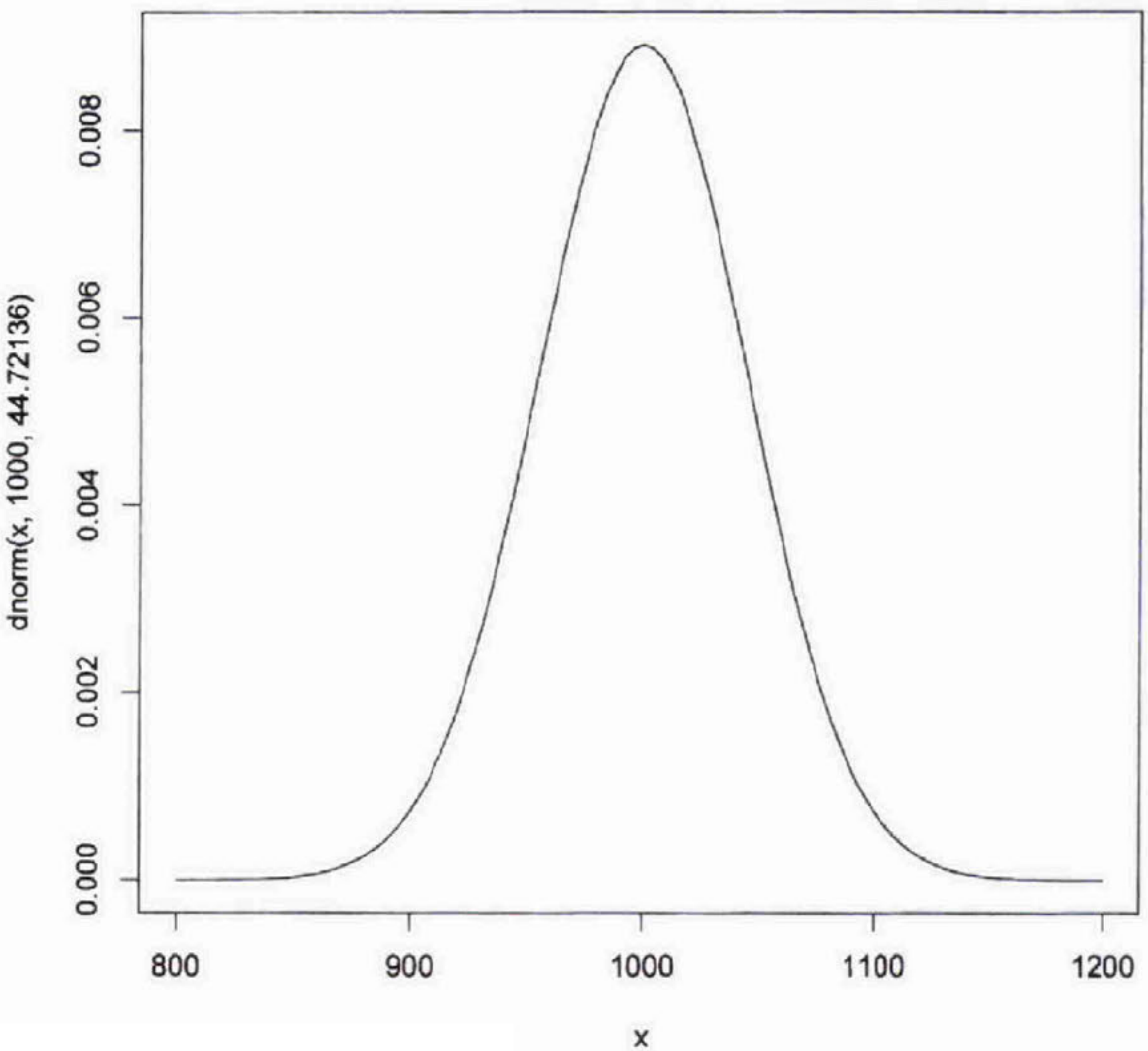
10,000 iid samples from $X \sim \chi^2(1000)$



pdf of $X \sim \chi^2(1000)$

$$\therefore X = \sum_{k=1}^{1000} X_k \overset{\text{CLT}}{\approx} N(1000, 2000)$$

for iid $X_k \sim \chi^2(1)$



pdf of $X \sim N(1000, 2000)$

$\therefore \sigma_X = \sqrt{2000} = 44.72136$

$\therefore$ Use in R: curve(dnorm(x, 1000, 44.72136), 800, 1200)